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NUMERICAL SOLUTIONS OF SHOCK CURVES OF HYPERBOLIC 1-CONSERVATION LAWS

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Abstract—We consider a certain hyperbolic 1-conservation laws. Such laws appear in problems of traffic flow, flood waves, and chemical exchange processes, etc., as illustrated by Whitham. We present certain numerical methods that estimate the shock curves of such laws. These include (i) linear and quadratic interpolations to approximate the left and right states of a shock curve, (ii) predictor–corrector methods to solve shock differential equations.

1. INTRODUCTION

The hyperbolic 1-conservation law under consideration is

$$\rho_t(x, t) + Q_x(\rho(x, t)) = 0, \qquad \rho(x, 0) = f(x) \tag{1}$$

for $x \in R = (-\infty, \infty)$, where $Q(\rho)$ is a function of ρ. In general the solution of (1) develops a singularity or breaks down in the sense that its first order derivatives become unbounded at finite time. Afterwards multiple solutions appear. Analytically we can extend the original solution to a physical reasonable one beyond the breakdown time by inserting shock curves. See, e.g., Whitham [1, Sec. 2.9] and Chang [2]. Numerical methods, such as finite difference method, have been introduced to estimate the positions of shock curves. See, e.g., Forsythe and Wasow [3, Sec. 10.2]. Lax and Wendroff [4], Ames [5, Sec. 5.2], Salas [6], de Neef and Hechtman [7], and Le Roux [8]. These methods assumed the existence of shock curves and aimed at systems of equations. They disregarded the possibility of nonexistence of shock curves for a single equation such as (1), even though there is a singularity. As was noted in Chang [2], a shock curve can be described by the initial value problem of a first order nonlinear nonautonomous ordinary differential equation. We call such an equation a shock differential equation (SDE). A SDE depends on the states on the two sides of the shock curve. For some examples these two states cannot be clearly identified in the sense that there are two choices for each state. One such example is Example 2 in Sec. 4 of Chang [2]. In Chang [2], we presented some conditions which imply the uniqueness of these two states and introduced a method to estimate them. In this paper, assuming these conditions [(3) and (8)] and using the above method we present a numerical method to solve a SDE (12).

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We first derive a SDE. Let $c(\rho) = Q'(\rho)$. Then (1) becomes

$$\rho_t(x, t) + c(\rho(x, t)\rho_x(x, t) = 0, \quad \rho(x, 0) = f(x). \tag{2}$$

For $x \in R$, let $F(\xi) = c(f(\xi))$, $G(\xi, t) = \xi + F(\xi)t$, $-m = \text{g.l.b.}\ \{F'(\xi) : \xi \in R\}$, $\mathscr{B} = \{\xi \in R : F'(\xi) = -m\}$, and $\mathscr{Z} = \{\xi \in R : F''(\xi) = 0\}$. We assume that

$$Q \in C^3(R); f \in C^2(R); 0 < m < \infty; \mathscr{B} \text{ is nonempty and } \mathscr{Z} \text{ is an isolated set.} \tag{3}$$

For any ξ, let the line C_ξ denote the characteristic curve $x = G(\xi, t)$ for (2). It can be verified easily that along a line C_ξ the function

$$\rho(x, t) = f(\xi) \tag{4}$$

is a solution of (2). Nevertheless, for two distinct values ξ_1 and ξ_2 such that $f(\xi_1) \neq f(\xi_2)$, if the two lines C_{ξ_1} and C_{ξ_2} meet at a point (x', t'), then the solution defined in (4) has multiple values at (x', t'). This, as mentioned before, is interpreted as the occurrence of a singularity. For a fixed ξ, differentiating (4) along the line C_ξ and using (2) we find

$$\rho_t = -\frac{F(\xi)f'(\xi)}{1 + F'(\xi)t}, \qquad \rho_x = \frac{f'(\xi)}{1 + F'(\xi)t}. \tag{5}$$

Let $t_B = 1/m$. Equations (5) imply that for any element ξ_B in $\mathscr{B}$, the envelop with implicit equations

$$x = G(\xi, t), \quad 0 = 1 + F'(\xi)t, \tag{6}$$

which has a cusp at (x_B, t_B), where $x_B = G(\xi_B, t_B)$, is a curve of singularity. For a fixed ξ_B in $\mathscr{B}$, $F'(\xi_B) = -m$. By (3), the set $\mathscr{Z}$ is an isolated set. For small $t - t_B > 0$ there are precisely two numbers $\xi^j(t)$ $(j = 1, 2)$, $\xi^1(t) < \xi_B < \xi^2(t)$ and $F''(\xi^1(t)) < 0 < F''(\xi^2(t))$, such that

$$F'(\xi^1(t)) = F'(\xi^2(t)) = -\frac{1}{t} \tag{7}$$

Differentiating (7) yields $(-1)^j \xi^{j\prime}(t) = (-1)^j [t^2 F''(\xi^j(t))]^{-1} > 0$ $(j = 1, 2)$. Set $\xi^j(t_B) = \xi_B$ and $\phi^j(t) = G(\xi^j(t), t)$, $(j = 1, 2)$. Then $\phi^{1\prime}(t_B) = \phi^{2\prime}(t_B)$ and $\phi^{1\prime\prime}(t) - \phi^{2\prime\prime}(t) = [t^3 F''(\xi^2(t))]^{-1} - [t^3 F''(\xi^1(t))]^{-1} > 0$ for small $t - t_B > 0$. Thus, $\phi^1(t) > \phi^2(t)$ for small $t - t_B > 0$. According to (6) and (7) the curves $x = \phi^1(t)$ and $\phi^2(t)$ form the branches of the curve of singularity which has a cusp at (x_B, t_B), where $x_B = G(\xi_B, t_B)$. We assume the following conditions:

$$\begin{aligned} &\text{Either } \lim_{\xi \to -\infty} F'(\xi) > -m \text{ or } F(\xi) \text{ is bounded on } (-\infty, a] \\ &\text{Either } \lim_{\xi \to \infty} F'(\xi) > -m \text{ or } F(\xi) \text{ is bounded on } [b, \infty). \end{aligned} \tag{8}$$

The numbers a and b in (8) are some constants. As was noted in Remark 1 of Chang [2], the following lemma holds.

Lemma 1. For sufficiently small τ, the following hold.

(i) For all $0 < t - t_B < \tau$, there is precisely one number $\xi^1_+(t)$, $\xi^1_+(t) > \xi^1(t)$, such that $\phi^1(t) = G(\xi^1_+(t), t)$.

(ii) For $0 < t - t_B < \tau$, there is precisely one number $\xi_-^2(t)$, $\xi_-^2(t) < \xi^2(t)$, such that $\emptyset^2(t) = G(\xi_-^2(t), t)$.

(iii) For any point in $\mathring{D}_\tau$, the interior of $D_\tau = \{(x, t): 0 \le t - t_B \le \tau; \emptyset^2(t) \le x \le \emptyset^1(t)\}$, there are precisely three numbers $\xi_-(x, t)$, $\xi(x, t)$, and $\xi_+(x, t)$, $\xi_-(x, t) < \xi(x, t) < \xi_+(x, t)$, such that $x = G(\xi_-(x, t), t) = G(\xi(x, t), t) = G(\xi_+(x, t), t)$.

We shall present an algorithm to estimate the functions $\xi^1(t)$, $\xi^2(t)$, $\xi_-(x, t)$, and $\xi_+(x, t)$ in Sec. 2. Suppose there is a smooth shock curve $x = s(t)$ lying in $\mathring{D}_\tau \cup \{(x_B, t_B)\}$ and originating at (x_B, t_B). Let

$$\rho_-(x, t) = f(\xi_-(x, t)), \qquad \rho_+(x, t) = f(\xi_+(x, t)) \tag{9}$$

for $(x, t) \in \mathring{D}_\tau$. According to (4) and Lemma 1, the functions $\rho_-(x, t)$ and $\rho_+(x, t)$ are the states of ρ on the left and right sides of $x = s(t)$. The Rankine–Hugoniot condition

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \frac{Q(\rho_-(s(t), t)) - Q(\rho_+(s(t), t))}{\rho_-(s(t), t) - \rho_+(s(t), t)}$$

for $t > t_B$ holds (see, e.g., Whitham [1, Sec. 2.3]). It follows from (9) that the shock curve $x = s(t)$ satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{Q(f(\xi_-(x, t)) - f(\xi_+(x, t))}{f(\xi_-(x, t)) - f(\xi_+(x, t))}, \qquad x(t_B) = x_B. \tag{10}$$

Since $c'(f(\xi_B))f'(\xi_B) = F'(\xi_B) = -m < 0$, there is a number $\epsilon > 0$ such that $c'(\delta) \neq 0$ for $f(\xi_B) - \epsilon < \delta < f(\xi_B) + \epsilon$ and $f'(\xi) \neq 0$ for $\xi_B - \epsilon < \xi < \xi_B + \epsilon$. By the mean value theorem, for any pair of numbers ξ, η between $\xi_B - \epsilon$ and $\xi_B + \epsilon$, there is a unique number $H(f(\xi), f(\eta))$, which is expressed as a function of $f(\xi)$ and $f(\eta)$, between $f(\xi)$ and $f(\eta)$ such that

$$Q(f(\xi)) - Q(f(\eta)) = c(H(f(\xi), f(\eta)))(f(\xi) - f(\eta)). \tag{11}$$

Substituting (11) into (10) gives the initial value problem of the shock differential equation (SDE)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(H(f(\xi_-(x, t)), f(\xi_+(x, t))), \qquad x(t_B) = x_B. \tag{12}$$

From now on by a SDE we mean either the differential equation in (12) or the initial value problem (12). As we observed in Remark 1 of Chang [2], we can prove the following theorem.

THEOREM 1. If conditions (3) and (8) hold, then the SDE (12) has a unique solution.

2. ALGORITHM OF ESTIMATING $\xi^1(t)$, $\xi^2(t)$, $\xi_-(x, t)$, and $\xi_+(x, t)$

Although the algorithm here is analogous to some for systems of equations (see, e.g., Forsythe and Wasow [3, Sec. 10.2]), it is unique in the sense that under condition (8) there is now precisely one left state $\xi_-(x, t)$ and precisely one right state $\xi_+(x, t)$ with no ambiguous terms. (See Chang [2]).

Let ξ_B be an element in $\mathscr{B}$. For a fixed $t > t_B$ such that $t - t_B$ is small, using the arguments in Secs. 2 and 4 of Chang [2], we can describe the local graph of the function $x = G(\xi, t)$

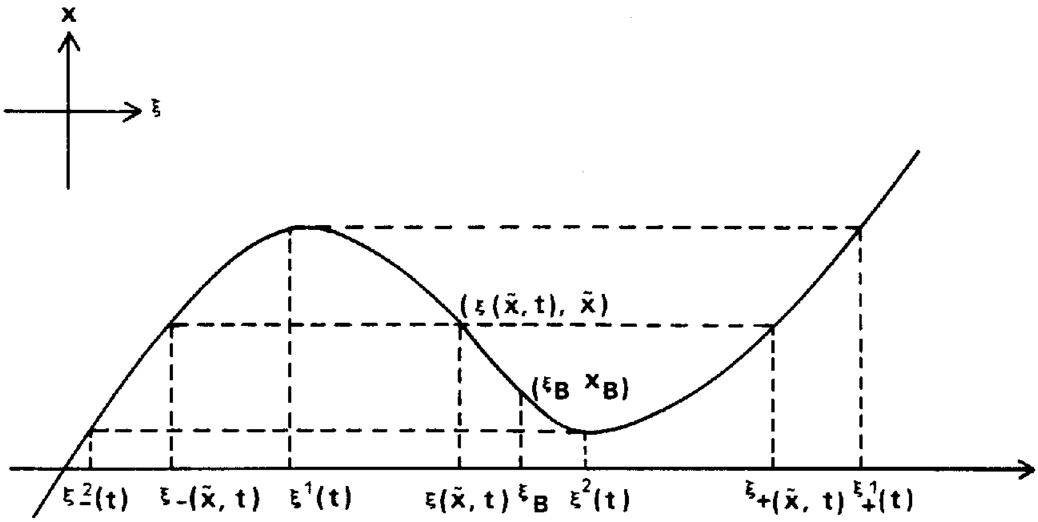

Fig. 1. Local graph of the function $x = G(\xi, t)$ of ξ about ξ_B.

of ξ about ξ_B as that in Fig. 1. According to Lemma 1 we have

Lemma 2. For $0 < t - t_B < \tau$, where τ is sufficiently small, the following hold:

(i) $G(\xi^1(t), t) = G(\xi^1_+(t), t) =$ a relative maximum of $G(\xi, t)$ for a unique number $\xi^1_+(t)$, $\xi^1_+(t) > \xi^1(t)$.

(ii) $G(\xi^2(t), t) = G(\xi^2_-(t), t) =$ a relative minimum of $G(\xi, t)$ for a unique number $\xi^2_-(t)$, $\xi^2_-(t) < \xi^2(t)$.

(iii) For $G(\xi^1_+(t), t) < x < G(\xi^2_-(t), t)$, there is a unique $\xi_-(x, t) < \xi^1(t)$ and a unique $\xi_+(x, t) > \xi^2(t)$ such that $x = G(\xi_-(x, t), t) = G(\xi_+(x, t), t)$.

We present the algorithm as follows.

(A) We divide an interval $\alpha \le \xi \le \beta$ about ξ_B into n subintervals of equal length: $\alpha = \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_{n+1} = \beta$, where

$$\xi_k = \xi_1 + (k-1)\left(\frac{\beta - \alpha}{n}\right)$$

for $1 \le k \le n + 1$.

(B) Evaluate $B(k) = G(\xi_k, t)$, $1 \le k \le n + 1$.

(C) For $1 \le k \le n - 2$, compare $B(k)$, $B(k+1)$, and $B(k+2)$. If $B(k) \le B(k+1)$ and $B(k+1) \ge B(k+2)$, then by Lemma 2(i),

$$\xi^1(t) \approx \xi_{k+1}. \tag{13}$$

If $B(k) \ge B(k+1)$ and $B(k+1) \le B(k+2)$, then by Lemma 2(ii),

$$\xi^2(t) \approx \xi_{k+1}. \tag{14}$$

(D) For $2 \le k \le n$, compare x with $B(k-1)$ and $B(k)$. If $B(k-1) \le x \le B(k)$ for the first time, then by Lemma 2(iii), using either linear interpolation

$$GL(w) = \frac{(w - B(k-1))}{B(k) - B(k-1)}(\xi_k - \xi_{k-1}) + \xi_{k-1} \tag{15}$$

or quadratic interpolation

$$GQ(w) = B(k-1)\frac{(w-\xi_k)(w-\xi_{k+1})}{(\xi_{k-1}-\xi_k)(\xi_{k-1}-\xi_{k+1})} + B(k)\frac{(w-\xi_{k+1})(w-\xi_{k-1})}{(\xi_k-\xi_{k+1})(\xi_k-\xi_{k-1})} + B(k+1)\frac{(w-\xi_{k-1})(w-\xi_k)}{(\xi_{k+1}-\xi_{k-1})(\xi_{k+1}-\xi_k)}, \tag{16}$$

we have

$$\xi_-(x,t) \approx GL(x) \tag{17a}$$

$$\xi_-(x,t) \approx GQ(x) \tag{17b}$$

If $B(k-1) \le x \le B(k)$ for the second time for a larger value of k, then by Lemma 2(iii), using (15) and (16) we have

$$\xi_+(x,t) \approx GL(x) \tag{18a}$$

$$\xi_+(x,t) \approx GQ(x). \tag{18b}$$

We can improve estimations (13), (14), (17), and (18) to be as precise as possible by choosing the integer n as large as possible.

For an example of traffic flow in Sec. 4, we wrote two computer programs to estimate $\xi^1(t)$ and $\xi^2(t)$ by (13) and (14). Program 1 was written in FORTRAN 77 and run on a CDC NOS system with results listed in Table 1. Program 2 was written in FORTRAN IV and run on an IBM 360/370 system with results listed in Table 2. In Sec. 4, we shall also use estimations (17) and (18).

3. THE PREDICTOR–CORRECTOR METHOD

We briefly introduce the method as follows (see, e.g., Conte and de Boor [9, Sec. 6.7]). We consider the initial value problem of a first-order ordinary differential equation

$$x'(t) = g(t,x), \qquad x(t_0) = x_0. \tag{19}$$

Integrating both sides of (19) from a number t_n to a number t_{n+1} and applying the Fundamental Theorem of Calculus yields

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} g(t, x(t))\,\mathrm{d}t. \tag{20}$$

Using the trapezoidal rule to approximate the integral in (20), we obtain

$$x(t_{n+1}) \approx x(t_n) + \frac{h}{2}[g(t_n, x(t_n)) + g(t_{n+1}, x(t_{n+1}))],$$

where $h = t_{n+1} - t_n$. The iteration scheme for (20) is then

$$x_{n+1} = x_n + \frac{h}{2}[g(t_n, x_n) + g(t_{n+1}, x_{n+1})].$$

Table 1. $C(K, I) = (K - 51) \cdot (0.05)\pi (1 \le K \le 120)$, $t_B \approx 0.10070063E+00$, $T(I) = t_B + 0.01I$, $B(K, I) = G(C(K, I), T(I))$

	Relative Maximum		Relative Minimum		Relative Maximum		Relative Minimum		Relative Maximum		Relative Minimum	
I	Value	K	Value	K	Value	K	Value	K	Value	K	Value	K
1	−0.53979181E+01	16	−0.54451930E+01	20	0.88526717E+00	56	0.83799234E+00	60	0.71684525E+01	96	0.71211776E+01	100
2	−0.53755252E+01	15	−0.55092061E+01	21	0.90766009E+00	55	0.77397925E+00	61	0.71908454E+01	95	0.70571646E+1	101
3	−0.53491166E+01	14	−0.55874408E+01	22	0.93406871E+00	54	0.69574452E+00	62	0.72172540E+01	94	0.69789298E+01	102
5	−0.52782937E+01	14	−0.57662665E+01	24	0.10048916E+01	54	0.51691878E+00	64	0.72880769E+01	94	0.68001041E+01	104
10	−0.50661426E+01	13	−0.62984543E+01	25	0.12170427E+01	53	−0.15268981E−01	65	0.75002280E+01	93	0.62679163E+01	105
20	−0.45895058E+01	12	−0.75118966E+01	27	0.16936795E+01	52	−0.12287112E+01	67	0.79768649E+01	92	0.50544741E+01	107

Table 2. $t_B \approx 0.10070056E+00$

	Relative Maximum		Relative Minimum		Relative Maximum		Relative Minimum		Relative Maximum		Relative Minimum	
I	Value	K	Value	K	Value	K	Value	K	Value	K	Value	K
1	−0.53979177E+01	16	−0.54451904E+01	20	0.88526833E+00	56	0.83799362E+00	60	0.71684532E+01	96	0.71211796E+01	100
2	−0.53755236E+01	15	−0.55092030E+01	21	0.90766007E+00	55	0.77398223E+00	61	0.71908455E+01	95	0.70571671E+01	101
3	−0.53491144E+01	14	−0.55874376E+01	22	0.93407035E+00	54	0.69574738E+00	62	0.72172546E+01	94	0.69789333E+01	102
5	−0.52782917E+01	14	−0.57662630E+01	24	0.10048933E+01	54	0.51692295E+00	64	0.72880783E+01	94	0.68001070E+01	104
10	−0.50661421E+01	13	−0.62984505E+01	25	0.12170448E+01	53	−0.15264511E−01	65	0.75002298E+01	93	0.62679214E+01	105
20	−0.45895052E+01	12	−0.75118914E+01	27	0.16936817E+01	52	−0.12287064E+01	67	0.79768639E+01	92	0.50544796E+01	107

Further, we adopt the inner iteration scheme:

$$x_{n+1}^{(0)} = x_n + hg(t_n, x_n), \tag{21}$$

$$x_{n+1}^{(1)} = x_n + \frac{h}{2}[g(t_n, x_n) + g(t_{n+1}, x_{n+1}^{(0)})], \tag{22}$$

$$\cdots$$

$$x_{n+1}^{(j)} = x_n + \frac{h}{2}[g(t_n, x_n) + g(t_{n+1}, x_{n+1}^{(j-1)})]. \tag{23}$$

4. APPLICATION TO TRAFFIC FLOW

For a certain model of traffic flow

$$Q(\rho) = A\rho \log \frac{\rho_{\max}}{\rho} \tag{24}$$

$$c(\rho) = Q'(\rho) = -A \log\left(\frac{\rho e}{\rho_{\max}}\right), \tag{25}$$

where A and $\rho_{\max}$ are constants. See Greenberg [10] and Whitham [1, Sec. 3.1]. We choose the initial function $f(x)$ such that (3) and (8) hold. The function $H(\rho_1, \rho_2)$ defined in (11) is

$$H(\rho_1, \rho_2) = \exp\left(\frac{\rho_1 \log \rho_1 - \rho_2 \log \rho_2}{\rho_1 - \rho_2} - \log \rho_{\max}\right).$$

Then,

$$c(H(\rho_1, \rho_2)) = -A\left(\frac{\rho_1 \log \rho_1 - \rho_2 \log \rho_2}{\rho_1 - \rho_2} - \log \rho_{\max}\right).$$

The SDE (12) is then

$$\frac{dx}{dt} = g(t, x), \; x(t_B) = x_B. \tag{26}$$

In (26), ξ_B is an element of $\mathscr{B}$, $t_B = -1/F'(\xi_B)$, $x_B = G(\xi_B, t_B)$, and

$$\begin{aligned} g(t,x) &= c(H(f(\xi_-(x,t)), f(\xi_+(x,t)))) \\ &= -A\left[\frac{f(\xi_-(x,t))\log(f(\xi_-(x,t))) - f(\xi_+(x,t))\log(f(\xi_+(x,t)))}{f(\xi_-(x,t)) - f(\xi_+(x,t))} - \log \rho_{\max}\right] \end{aligned}$$

for $(x, t) \in \mathring{D}_\tau$ with

$$\begin{aligned} g(t_B, x_B) &= \lim_{(x,t)\text{ in } \mathring{D}_\tau \to (x_B, t_B)} c(H(f(\xi_-(x,t)), f(\xi_+(x,t)))) \\ &= c(f(\xi_B)) = -A \log\left(\frac{f(\xi_B)e}{\rho_{\max}}\right), \end{aligned}$$

where the region D_τ is as described in Lemma 1. We choose the number h in (21), (22), (23) to be $h = 0.01$. We set $x_1 = x_B$, $t_1 = t_B$, and $t_{n+1} = nh + t_B = 0.01n + t_B$. We let L denote the total number of inner iterations. For $n = 1$, Eqs. (21) and (23) become

$$\begin{aligned} x_2^{(0)} &= x_B + 0.01 g(t_B, x_B) \\ &= x_B - 0.01 A Z_0; \end{aligned} \tag{27}$$

$$\begin{aligned} x_2^{(j)} &= x_B + 0.005[-AZ_0 + g(t_1, x_2^{(j-1)})] \\ &= x_B - 0.005A\left[Z_0 + \frac{E_1(t_2, x_2^{(j-1)}) \log E_1(t_2, x_2^{(j-1)}) - E_2(t_2, x_2^{(j-1)}) \log E_2(t_2, x_2^{(j-1)})}{E_1(t_2, x_2^{(j-1)}) - E_2(t_2, x_2^{(j-1)})}\right. \\ &\quad \left. - \log \rho_{\max}\right] (1 \le j \le L), \end{aligned} \tag{28}$$

where $Z_0 = \log(f(\xi_B)e/\rho_{\max})$, $E_1(t, x) = f(\xi_-(x, t))$, and $E_2(t, x) = f(\xi_+(x, t))$. For $n > 1$, Eqs. (21) and (23) are

$$\begin{aligned} x_{n+1}^{(0)} &= x_n + 0.01 g(t_n, x_n^{(L)}) \\ &= x_n - 0.01 A Z_0; \end{aligned} \tag{29}$$

$$\begin{aligned} x_{n+1}^{(j)} &= x_n + 0.005[-AZ_0 + g(t_{n+1}, x_{n+1}^{(j-1)})] \\ &= x_n - 0.005\left[Z_0 + \frac{E_1(t_{n+1}, x_{n+1}^{(j-1)}) \log E_1(t_{n+1}, x_{n+1}^{(j-1)}) - E_2(t_{n+1}, x_{n+1}^{(j-1)}) \log E_2(t_{n+1}, x_{n+1}^{(j-1)})}{E_1(t_{n+1}, x_{n+1}^{(j-1)}) - E_2(t_{n+1}, x_{n+1}^{(j-1)})}\right. \\ &\quad \left. - \log \rho_{\max}\right] (1 \le j \le L), \end{aligned} \tag{30}$$

where

$$Z_0 = \frac{E_1(t_n, x_n^{(L)}) \log E_1(t_n, x_n^{(L)}) - E_2(t_n, x_n^{(L)}) \log E_2(t_n, x_n^{(L)})}{E_1(t_n, x_n^{(L)}) - E_2(t_n, x_n^{(L)})} - \log \rho_{\max},$$

$E_1(t, x) = f(\xi_-(x, t))$ and $E_2(t, x) = f(\xi_+(x, t))$. We set the initial profile in (1) as $\rho(x, 0) = f(x) = \delta(2 - \cos x)\rho_{\max}$ for $x \in R$, where δ is a control parameter. Then $F(\xi) = c(f(\xi)) = -A \log[(2 - \cos \xi)e\delta]$ and $F'(\xi) = -A \sin \xi/(2 - \cos \xi)$. Now $F'(k\pi + \pi/3) = \min\{F'(\xi) : \xi \in R\}$, where k is any integer. We can verify easily that conditions (3) and (8) hold. Choose $\xi_B = \pi/3$. Then $t_B = -1/F'(\pi/3) = (2 - \cos(\pi/3))/(A \sin(\pi/3))$. For $A = 17.2$ as considered in Greenberg [10] and Whitham [1, Sec. 3.1], $t_B \approx 0.10070$ and $x_B = G(\xi_B, t_B) \approx 0.88069$. In addition to the shock curve originating at (x_B, t_B), we also attempt to estimate the flow velocity

$$V(\rho) = \frac{Q(\rho)}{\rho} = A \log \frac{\rho_{\max}}{\rho}. \tag{31}$$

If $x = s(t)$ describes the shock curve, by (4) and (31) we have that

$$\begin{aligned} \mathrm{VLT}(s(t), t) &= A \log \frac{\rho_{\max}}{f(\xi_-(s(t), t))} \\ &= -A \log(2 - \cos(\xi_-(s(t), t))) \end{aligned} \tag{32}$$

$$\begin{aligned} \mathrm{VRT}(s(t), t) &= A \log \frac{\rho_{\max}}{f(\xi_+(s(t), t))} \\ &= -A \log(2 - \cos(\xi_+(s(t), t))) \end{aligned} \tag{33}$$

denote the flow velocities to the left and the right of the shock curve, respectively.

Table 3. $x_1 = x_B \approx 0.88069241\mathrm{E}+00$, $t_1 = t_B = 0.10070063\mathrm{E}+00$, $t_{n+1} = t_1 + 0.01n$

n	j	t_{n+1}	$x^{(j)}_{n+1}$	$\xi_-(x^{(j)}_{n+1}, t_{n+1})$	$\xi_+(x^{(j)}_{n+1}, t_{n+1})$	VLT (mph)	VRT (mph)
1	0	0.11070063E + 00	0.86415774E + 00	0.47797318E + 00	0.17890257E + 01	0.20693480E + 02	0.88305373E + 01
	1		0.85685270E + 00	0.45157987E + 00	0.17463811E + 01	0.20877427E + 02	0.91581397E + 01
	2		0.85821238E + 00	0.45576092E + 00	0.17543185E + 01	0.20848805E + 02	0.90964815E + 01
	3		0.85796818E + 00	0.45501000E + 00	0.17528929E + 01	0.20853960E + 02	0.91075325E + 01
	4		0.85801200E + 00	0.45514475E + 00	0.17531487E + 01	0.20853035E + 02	0.91055487E + 01
	5		0.85800414E + 00	0.45512056E + 00	0.17531028E + 01	0.20853201E + 02	0.91059047E + 01
	6		0.85800555E + 00	0.45512490E + 00	0.17531110E + 01	0.20853171E + 02	0.91058408E + 01
	7		0.85800529E + 00	0.45512412E + 00	0.17531096E + 01	0.20853175E + 02	0.91058523E + 01
	8		0.85800534E + 00	0.45512426E + 00	0.17531098E + 01	0.20853176E + 02	0.91058502E + 01
	9		0.85800533E + 00	0.45512424E + 00	0.17531098E + 01	0.20853176E + 02	0.91058506E + 01
	10		0.85800533E + 00	0.45512424E + 00	0.17531098E + 01	0.20853176E + 02	0.91058505E + 01
2	0	0.12070063E + 00	0.82916584E + 00	0.26137992E + 00	0.20709984E + 01	0.21946034E + 02	0.69012403E + 01
	1		0.82428926E + 00	0.25218014E + 00	0.20546546E + 01	0.21984936E + 02	0.70014527E + 01
	2		0.82468054E + 00	0.25291814E + 00	0.20559659E + 01	0.21981862E + 02	0.69933581E + 01
	3		0.82464906E + 00	0.25285877E + 00	0.20558604E + 01	0.21982110E + 02	0.69940089E + 01
	4		0.82465160E + 00	0.25286355E + 00	0.20558689E + 01	0.21982090E + 02	0.69939566E + 01
	5		0.82465139E + 00	0.25286317E + 00	0.20558683E + 01	0.21982091E + 02	0.69939608E + 01
	6		0.82465141E + 00	0.25286320E + 00	0.20558683E + 01	0.21982091E + 02	0.69939604E + 01
	10	0.12070063E + 00	0.82465141E + 00	0.25286319E + 00	0.20558683E + 01	0.21982091E + 02	0.69939605E + 01
3	0	0.13070063E + 00	0.78678305E + 00	0.11078745E + 00	0.22828745E + 01	0.22415408E + 02	0.57359992E + 01
	5		0.78361639E + 00	0.10694850E + 00	0.22739509E + 01	0.22422540E + 02	0.57800123E + 01
	10		0.78361639E + 00	0.10694850E + 00	0.22739509E + 01	0.22422540E + 02	0.57800123E + 01
5	0	0.15070063E + 00	0.68776645E + 00	−0.94882509E − 01	0.26081707E + 01	0.22443342E + 02	0.44399547E + 01
	10		0.68587825E + 00	−0.96453498E − 01	0.26045758E + 01	0.22440772E + 02	0.44509806E + 01
10	0	0.20070063E + 00	0.39311854E + 00	−0.40213459E + 00	0.31153808E + 01	0.21200429E + 02	0.36263713E + 01
	10		0.39254884E + 00	−0.40239082E + 00	0.31147545E + 01	0.21198831E + 02	0.36264666E + 01
20	0	0.30070063E + 00	−0.26207496E + 00	−0.71733978E + 00	0.36209861E + 01	0.18731481E + 02	0.42831479E + 01
	10		−0.26213807E + 00	−0.71735682E + 00	0.36209537E + 01	0.18731326E + 02	0.42830589E + 01

Table 4. $x_1 = x_B \approx 0.8806924099426869\text{D}+00$, $t_1 = t_B \approx 0.1007006283470277\text{D}+00$, $t_{n+1} = t_1 + 0.01n$

n	j	$x_{n+1}^{(j)}$	VLT (mph)	VRT (mph)
1	0	0.8641577423852897D + 00	0.2070873523764511D + 02	0.8752792669799826D + 01
	1	0.8566436607004165D + 00	0.2092574967401790D + 02	0.9128206793671807D + 01
	2	0.8582144546041491D + 00	0.2088465940718852D + 02	0.9042625539788935D + 01
	3	0.8578726918140851D + 00	0.2089373538049847D + 02	0.9060872352856164D + 01
	4	0.8579461767794184D + 00	0.2089179041187883D + 02	0.9056932085158284D + 01
	5	0.8579303374424577D + 00	0.2089220993903075D + 02	0.9057780602126952D + 01
	6	0.8579337497115038D + 00	0.2089211957578645D + 02	0.9057597770608515D + 01
	7	0.8579330145282701D + 00	0.2089213904235867D + 02	0.9057637155172573D + 01
	8	0.8579331728994168D + 00	0.2089213484993375D + 02	0.9057628676226422D + 01
	9	0.8579331388013353D + 00	0.2089213575285661D + 02	0.9057630502182329D + 01
	10	0.8579331461445659D + 00	0.2089213555795254D + 02	0.9057630104449344D + 01
2	0	0.8289492829135502D + 00	0.2198766133267378D + 02	0.6877652942620980D + 01
	1	0.8241435347903450D + 00	0.2202600350037871D + 02	0.6990095825752201D + 01
	2	0.8245732633924812D + 00	0.2202267707471793D + 02	0.6979743140455394D + 01
	3	0.8245340729038279D + 00	0.2202298124861039D + 02	0.6980684751684177D + 01
	4	0.8245376404075421D + 00	0.2202295356584954D + 02	0.6980599017954504D + 01
	5	0.8245373156089546D + 00	0.2202295608676977D + 02	0.6980606822696033D + 01
	6	0.8245373451783760D + 00	0.2202295585748856D + 02	0.6980606109242842D + 01
	10	0.8245373427077710D + 00	0.2202295587630138D + 02	0.6980606170498921D + 01
3	0	0.7867295990651970D + 00	0.2242857728997700D + 02	0.5705938350474098D + 01
	5	0.7834585065881441D + 00	0.2243588344066368D + 02	0.5750454411187775D + 01
	10	0.7834585075100071D + 00	0.2243588343867639D + 02	0.5750454398222549D + 01
5	0	0.6873553644427235D + 00	0.2243263642736202D + 02	0.4427941857318271D + 01
	10	0.6854675342463139D + 00	0.2243003025597887D + 02	0.4438559859933253D + 01
10	0	0.3923292224095847D + 00	0.2117449038237701D + 02	0.3626192212786493D + 01
	10	0.3917528169350221D + 00	0.2117288826553382D + 02	0.3626278289134526D + 01
20	0	−0.2633221905436035D + 00	0.1871178793044651B + 02	0.4282838702181878D + 01
	10	−0.2633958302833125D + 00	0.1871160869166814D + 02	0.4282727941987101D + 01

As we mentioned at the end of Sec. 2, we wrote two computer programs to estimate $\xi^1(t)$ and $\xi^2(t)$ by (13) and (14) with input data $\rho_{\max} = 228$ and $\delta = 0.27$ and with results listed in Tables 1 and 2. Using the same input data, we also wrote two more programs. The first estimates the shock curve originating at (x_B, t_B) by algorithms (27), (28), (29), (30), and estimates VLT and VRT by (32) and (33), using the method of linear interpolation (17a) and (18a), with results listed in Table 3. The second performs the same task as the first does, using the method of quadratic interpolation (17b) and (18b), with results listed in Table 4. It also uses double precision type real variables for more precise estimations. Both programs were written in FORTRAN 77 and run on a CDC NOS system.

5. APPLICATIONS TO FLOOD WAVES AND CHEMICAL EXCHANGE PROCESS

For flood waves, system (1) can be written as

$$A_t(x, t) + Q_x(A(x, t)) = 0, \; A(x, 0) = f(x), \tag{34}$$

where $A(x, t)$ denotes the cross-sectional area of the river bed at position x along the river at time t. System (34) was established by Kleitz [1858, unpublished] and Seddon [11]. See also Whitham [1, Sec. 3.2].

If $Q(A) = bA^{n+1}$ as considered by Whitham [1], where b and n are positive constants, then

$$c(A) = Q'(A) = b(n+1)A^n$$
$$H(A_1, A_2) = \left[\frac{A_2^{n+1} - A_1^{n+1}}{(n+1)(A_2 - A_1)}\right]^{1/n}.$$

We choose the initial function $f(x)$ in (34) such that (3) and (8) hold. Given a breakdown point (x_B, t_B), by Lemma 1 the functions $\xi_-(x, t)$ and $\xi_+(x, t)$ exist in a region with (x_B, t_B) as a boundary point. By Theorem 1 the SDE (12) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b\frac{[\xi_+(x, t)]^{n+1} - [\xi_-(x, t)]^{n+1}}{\xi_+(x, t) - \xi_-(x, t)}, \qquad x(t_B) = x_B, \tag{35}$$

has a unique solution. The algorithms introduced in Secs. 2 and 3 can be applied to finding a numerical solution of (35).

For chemical exchange process the system (1) describes sedimentation in rivers, where $\rho(x, t)$ denotes the density of the field at position x and at time t. See Kynch [12] and Whitham [1, Sec. 3.4]. Suppose

$$c(\rho) = V - \frac{k_1k_2AB}{k_1k_2AB + [k_2B + (k_1 - k_2)\rho]^2} \tag{36}$$

as considered in Whitham [1], where V, A, B, k_1, and k_2 are constants. Then

$$Q(\rho) = \int c(\rho)\,\mathrm{d}\rho = V\rho - \frac{V\sqrt{k_1k_2AB}}{k_1 - k_2}\tan^{-1}\left[\frac{k_2B + (k_1 - k_2)\rho}{\sqrt{k_1k_2AB}}\right] + C, \tag{37}$$

where C is a constant. Substituting (36) and (37) into (11) we can find the function $H(\rho_1, \rho_2)$. Choose the initial function $f(x)$ in (1) such that (3) and (8) hold. Given a breakdown point (x_B, t_B), by Theorem 1 the SDE (12) has a unique solution. The algorithms introduced in Secs. 2 and 3 can again be applied to finding a numerical solution of (12).

6. BURGERS' EQUATION, UNIQUENESS, AND REGULARITY

The models discussed in Secs. 4 and 5 are only rough approximations to some nonlinear phenomena, which in some cases are too complicated to be described by a first-order equation such as that in (1). For example, in traffic flow the assumption that drivers follow a certain driving pattern is somewhat strict. Further, as can be observed, a shock region is encompassed by a layer with some degree of thickness. System (1) simplifies it to a shock curve with no thickness. Nevertheless, for some simpler cases, if one is interested to know the states outside a shock layer and adjacent to it, such as the flow velocities in a simple model of traffic flow, system (1) does provide a reasonable estimation.

If $Q'(\rho)$ is a one-to-one function, then by setting $v = Q'(\rho)$ we can transform (1) to $v_t(x, t) + v(x, t)v_x(x, t) = 0$, $v(x, 0) = V(x)$, which is the limiting case of the Burgers' equation

$$v_t(x, t) + v(x, t)v_x(x, t) = \nu v_{xx}(x, t), \qquad v(x, 0) = V(x)$$

of which the solution is known explicitly. See, e.g., Hopf [13], Cole [14], Lighthill [15], and Whitham [1, Chap. 4]. The algorithms introduced in Secs. 2 and 3 do not depend on whether the function $Q'(\rho)$ is one-to-one or not.

As was noted in Chang [2], the uniqueness of the analytical solution of the SDE (12) follows from the equal area method of Whitham [1, Sec. 2.9]. The functions $\xi_-(x, t)$ and $\xi_+(x, t)$ in Lemma 1 help us to construct the algorithms in Secs. 2 and 3, which avoid the difficulty of multiple solutions that we might encounter if we attempt to estimate a shock curve by solving (1) through the use of a finite difference method. As to extend the numerical solution of the SDE (12) to large time t, we need to take into consideration the interactions of shock curves.

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